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A Note on Rational Approximation to $(1-x)^{1/2}$

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According to a well-known result of S. N. Bernstein [1, p. 124] $(1-x)^{1/2}$ can be approximated uniformly on [0, 1] by polynomials of degree $\leqslant n$ with an error roughly $0.282/2(2^{1/2})\,n$. On the other hand it is obvious that $(1-x)^{1/2}$ cannot be approximated uniformly on [0, 1] arbitrarily closely by real polynomials having only nonnegative coefficients. Now it is natural to ask whether one can approximate $(1-x)^{1/2}$ on [0, 1] arbitrarily closely by rational functions having only real nonnegative coefficients. In this connection we prove here

THEOREM 1. *For all $n \geqslant 2$,*

$$\left\| (1-x)^{1/2} - \frac{1}{\sum_{k=0}^{n} \binom{2k}{k}\left(\frac{x}{4}\right)^k} \right\|_{L_\infty[0,1]} \leqslant 4\left(\frac{\log n}{n}\right)^{1/2}.$$

Proof. It is well known that

$$(1-x)^{-1/2} = \sum_{k=0}^{\infty} \binom{2k}{k}\left(\frac{x}{4}\right)^k \qquad \text{and} \qquad k^{1/2}\binom{2k}{k} < 4^k.$$

Hence, for $0 \leqslant x \leqslant (n - 2\log n)\,n^{-1}$,

$$0 \leqslant \frac{1}{\sum_{k=0}^{n} \binom{2k}{k}\left(\frac{x}{4}\right)^k} - (1-x)^{1/2} \leqslant \sum_{k=n+1}^{\infty} \binom{2k}{k}\left(\frac{x}{4}\right)^k \leqslant \frac{1}{n^{1/2}} \sum_{k=n+1}^{\infty} x^k \leqslant \frac{1}{n^{1/2}}.$$

The last inequality follows, e.g., by applying $\log(1-t) \leqslant -t$ to $t = 2n^{-1}\log n$. On the other hand, for $x \geqslant (n - 2\log n)/n$,

$$0 \leqslant \frac{1}{\sum_{k=0}^{n} \binom{2k}{k}\left(\frac{x}{4}\right)^k} - (1-x)^{1/2} \leqslant \frac{1}{\sum_{k=0}^{n} \binom{2k}{k}\left(\frac{x}{4}\right)^k}$$

$$\leqslant \frac{1}{\sum_{k=0}^{n} \binom{2k}{k}\left(\frac{n-2\log n}{4n}\right)^k} \leqslant 4\left(\frac{\log n}{n}\right)^{1/2},$$

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since

$$\sum_{k=0}^{n} \binom{2k}{k} \left(\frac{n - 2\log n}{4n}\right)^k = \left(\frac{n}{2\log n}\right)^{1/2} - \sum_{k=n+1}^{\infty} \binom{2k}{k} \left(\frac{n - 2\log n}{4n}\right)^k$$

$$\geqslant \left(\frac{n}{2\log n}\right)^{1/2} - \frac{1}{n^{1/2}} \geqslant \frac{1}{4}\left(\frac{n}{\log n}\right)^{1/2}.$$

Hence the result.

THEOREM 2. *For real polynomials $p(x)$, $q(x)$ of degrees at most $n \geqslant 12$, having only nonnegative coefficients,*

$$\left\| (1 - x)^{1/2} - \frac{p(x)}{q(x)} \right\|_{L_\infty[0,1]} \geqslant \frac{1}{4(n^{1/2})}.$$

Proof. Set

$$\epsilon = \left\| (1 - x)^{1/2} - \frac{p(x)}{q(x)} \right\|_{L_\infty[0,1]}.$$

Then

$$\epsilon^{-1} \leqslant \left\| \frac{q(1)}{p(1)} \right\|, \tag{1}$$

and on $0 \leqslant x \leqslant (n - 1)/n$,

$$\frac{p(x)}{q(x)} \geqslant (1 - x)^{1/2} - \epsilon \geqslant \frac{1}{n^{1/2}} - \epsilon. \tag{2}$$

If $1/n^{1/2} - \epsilon \leqslant 0$, then $\epsilon \geqslant 1/n^{1/2}$. Otherwise (3)

$$\max_{[0,(n-1)/n]} \left\| \frac{q(x)}{p(x)} \right\| \leqslant \frac{n^{1/2}}{1 - \epsilon(n)^{1/2}}. \tag{4}$$

It is easy to verify that

$$| q(1)| = \left| q\left(\frac{n-1}{n} \cdot \frac{n}{n-1}\right)\right| \leqslant \left(\frac{n}{n-1}\right)^n \left| q\left(\frac{n-1}{n}\right)\right| \leqslant 3 \left| q\left(\frac{n-1}{n}\right)\right|$$

and

$$\left| \frac{q(1)}{p(1)} \right| \leqslant \frac{3\,| q((n-1)/n)|}{| p((n-1)/n)|} \leqslant \frac{3(n^{1/2})}{1 - \epsilon(n^{1/2})}. \tag{5}$$

From (1) and (5),

$$\epsilon \geqslant \frac{1}{4(n^{1/2})}. \tag{6}$$

The desired conclusion follows from (3) and (6).

REFERENCE

1. S. N. BERNSTEIN, "Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réele," Gauthier-Villars, Paris, 1926.